Solutions to HW #4

1. a) $(1, 45°, 1)$

$x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad z = 1$

$= 1\cdot\cos 45°$ $\quad = 1\cdot\sin 45°$

$= \frac{1}{\sqrt{2}}$ $\quad = \frac{1}{\sqrt{2}}$

Hence the corresponding point is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right)$

$(2, \frac{\pi}{2}, -4)$

$x = 2\cos\frac{\pi}{2}$ $\quad y = 2\sin\frac{\pi}{2}$ $\quad z = -4$

$= 0$ $\quad = 2$

Hence the corresponding point in Cartesian coordinates is $(0, 2, -4)$

$(3, \frac{\pi}{6}, 0)$

$x = 3\cos\frac{\pi}{6}$ $\quad y = 3\sin\frac{\pi}{6}$ $\quad z = 0$

$= 3\frac{\sqrt{3}}{2}$ $\quad = \frac{3}{2}$

So the point is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0\right)$

b) $(2, 1, -2)$

$\rho = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = 3$

$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\varphi = \cos^{-1}\left(\frac{-2}{3}\right)$

Hence the point in spherical coordinates is $\left(3, \tan^{-1}\left(\frac{1}{2}\right), \cos^{-1}\left(\frac{-2}{3}\right)\right)$

$(0, 3, 4)$

$$\rho = \sqrt{0^2+3^2+4^2} = \sqrt{9+16} = 5$$

$$\theta = \frac{\pi}{2}$$

$$\varphi = \cos^{-1}\left(\frac{4}{5}\right)$$

So the point is $\left(5, \frac{\pi}{2}, \cos^{-1}\left(\frac{4}{5}\right)\right)$

$(-2\sqrt{3}, -2, 3)$

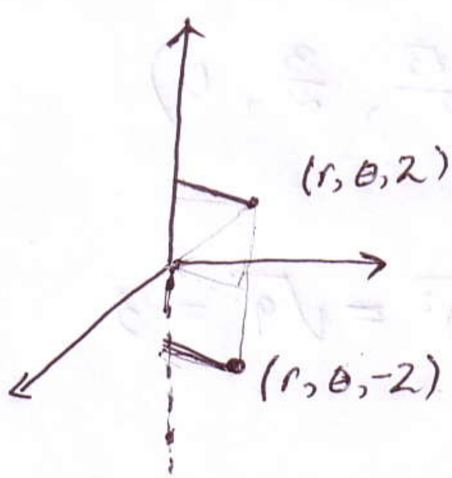
$$\rho = \sqrt{(-2\sqrt{3})^2+(-2)^2+3^2} = \sqrt{12+4+9} = 5$$

$$\theta = \pi + \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right) = \pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\varphi = \cos^{-1}\left(\frac{3}{5}\right)$$

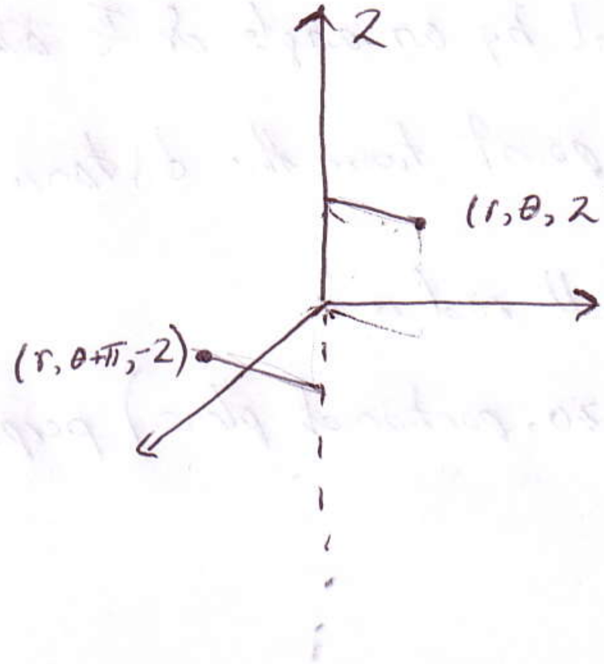
Hence the point is $\left(5, \frac{7\pi}{6}, \cos^{-1}\left(\frac{3}{5}\right)\right)$.

2. a)



This is reflection in the xy plane.

b)

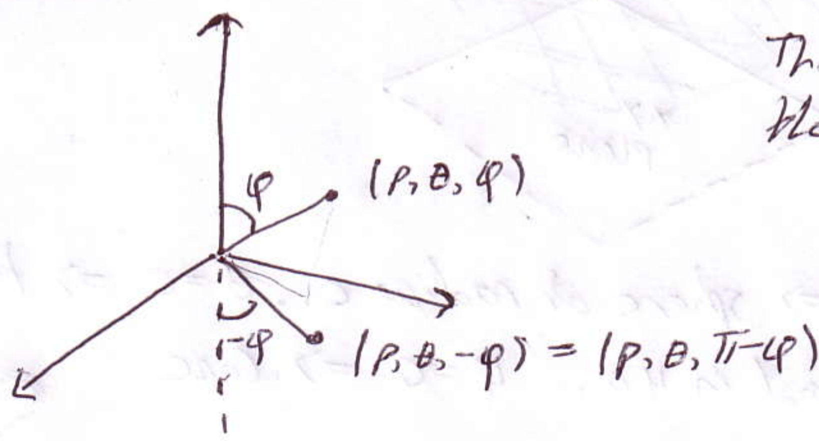


This is reflection in the xy plane and (followed by) ~~reflection~~ rotation by π. In other words, this is reflection through the origin.

c) $(r, \theta, z) \mapsto (-r, \theta - \frac{\pi}{4}, z)$
is a reflection in the z-axis followed by clockwise rotation by $\frac{\pi}{4}$.
You may think of $(-r, \theta - \frac{\pi}{4}, z)$ as $(r, \theta + \pi - \frac{\pi}{4}, z)$.

3. a) This is just a reflection in the z-axis.
Also it is the same thing as rotating each point by π (counterclockwise).

b)

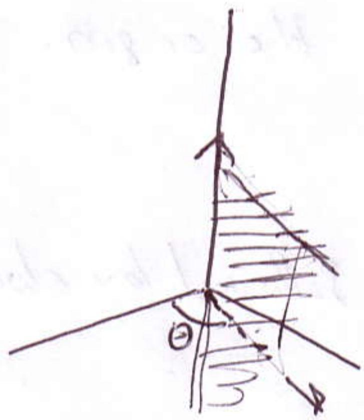


This is reflection in the xy plane

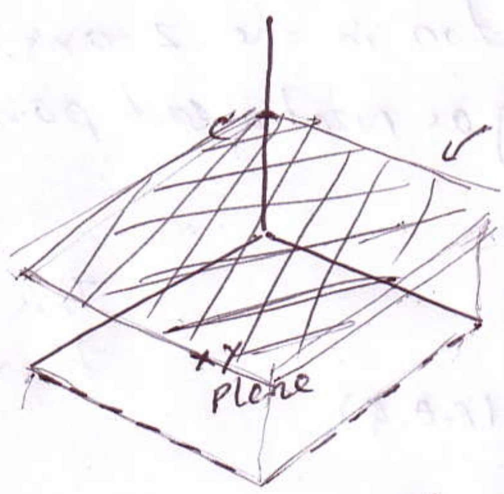
c) This rotates each point by an angle of $\frac{\pi}{2}$ about the z-axis and moves this point twice the distance from the origin.

4. a) $r = c \Rightarrow$ cylinder with radius c.

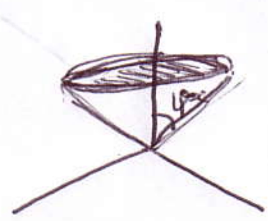
$\theta = c \Rightarrow$ plane (if $r \geq 0$, portion of plane) perpendicular to the xy plane.



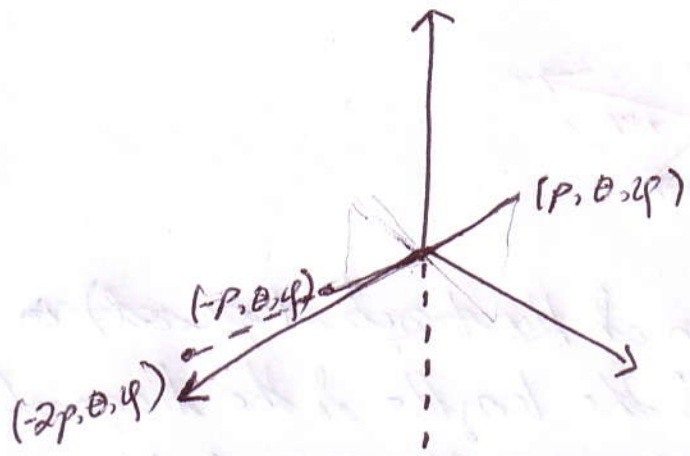
$z = c \Rightarrow$ a plane parallel to the xy plane.



b) $\rho = c \Rightarrow$ sphere of radius c., $\theta = c \Rightarrow$ the same plane featured in 4a. $\varphi = c \Rightarrow$ cone



5. the transformation $(p, \theta, \varphi) \mapsto (-p, \theta, \varphi)$ is a reflection through the origin of the xyz coordinate system.



the transformation $(p, \theta, \varphi) \mapsto (2p, \theta, \varphi)$ moves each point farther out from the origin.

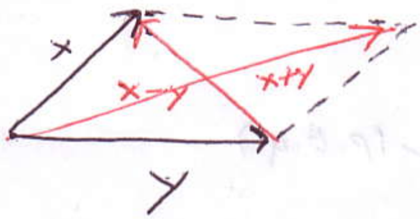
Its effect on a surface is to make it twice as large.

Transforming $(p, \theta, \varphi) \mapsto (2p, \theta, \varphi) \mapsto (-2p, \theta, \varphi)$ makes the surface twice as large and reflects it through the origin.

6. Since $r = \sqrt{x^2+y^2} \leq \sqrt{a^2} = a$, we see that $0 \leq r \leq a$. $x^2+y^2 \leq a^2$ is a disc of radius a. To move about this disc, we must be able to rotate by all angles $\theta \in [0, 2\pi]$. Hence $0 \leq \theta \leq 2\pi$. Finally, the problem states that $|z| \leq b$.

7. $(1,-1,0,2)\cdot(1,2,3,4) = 1-2+0+8 = 7$

8. a) This is known as the parallelogram law.



It states that the sum of ~~the lengths(squared) in~~ ~~squared~~ the squares of the lengths of the diagonals of a parallelogram is ~~two times the sum of~~ the sum of squares of its sides.

$$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) + (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) =$$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\vec{x}\cdot\vec{y} =$$

$$= 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$$

b) $\|\vec{x}+\vec{y}\|^2 - \|\vec{x}-\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\vec{x}\cdot\vec{y} -$

$$-(\|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\vec{x}\cdot\vec{y}) = 4\vec{x}\cdot\vec{y}.$$

9. $\|\vec{x}\| = \|x_1\vec{e}_1 + x_2\vec{e}_2 + \ldots + x_n\vec{e}_n\| \leq \sum_{i=1}^{n} \|x_i e_i\| = \sum_{i=1}^{n} |x_i|$

$$= \sum_{i=1}^{n} |x_i \cdot 1| = |(|x_1|, |x_2|, \ldots, |x_n|)\cdot(1,1,\ldots,1)| \leq$$

$$\leq \|(|x_1|, |x_2|, \ldots, |x_n|)\|\,\|(1,1,\ldots,1)\| = \sqrt{n}\,\|\vec{x}\|$$

10. a) $AC + D^T = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} =$

$= \begin{pmatrix} 2\cdot 10 + 3\cdot 1 \\ -1\cdot 10 + 0\cdot 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 23+2 \\ -10+5 \end{pmatrix} = \begin{pmatrix} 25 \\ -5 \end{pmatrix}$

b) $AB = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 6 & 7 & 1 \\ 0 & 4 & 2 \end{pmatrix} =$

$= \begin{pmatrix} 12 & 26 & 8 \\ -6 & -7 & -1 \end{pmatrix}$

c) BA is not defined.

d) $B^T = \begin{pmatrix} 6 & 0 \\ 7 & 4 \\ 1 & 2 \end{pmatrix}$

e) $B^TC = \begin{pmatrix} 6 & 0 \\ 7 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \end{pmatrix} = \begin{pmatrix} 60 \\ 74 \\ 12 \end{pmatrix}$